QED out of matter

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Abstract

The Wilsonian renormalization group implies that an arbitrary four dimensional field theory with an ultraviolet cutoff is equivalent to a theory which is renormalizable by power counting at energy scales much below the cutoff. This applies to any theory including those with non-renormalizable interactions as long as we fine-tune the mass parameters. We analyze two simple models with current-current interactions but without elementary gauge fields from this viewpoint. We show how to tune the parameters of the models so that they become equivalent to QED at energies much below the cutoff.

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1 Introduction

There seem to be two kinds of field theories: one good and the other bad. Good theories are renormalizable theories, such as ϕ^4 theory, QED, and QCD, which are well defined at all energy scales, and for which everything can be calculated in terms of a finite number of parameters. Bad theories are non-renormalizable theories, such as the four-Fermi weak theory and non-linear sigma model, which are well defined only below a finite cutoff energy (or momentum), and for which more and more parameters must be introduced as the order of approximations increases. Theories can be classified using the standard power counting rule: if the lagrangian of a theory contains only fields with dimension four or less, it is renormalizable, and otherwise it is non-renormalizable. This classification into renormalizable and non-renormalizable theories is so simple and quite popular, but we know it is wrong.

The physical meaning of renormalization and renormalizability cannot be understood without the renormalization group (to be abbreviated as RG) introduced by K. G. Wilson.[1] Using the renormalization group we also distinguish two classes of theories, but this time one is the class of ϕ^4-like theories, and the other is that of QCD-like theories.

A ϕ^4-like theory or a non-asymptotic free theory is characterized by an IR fixed point of the renormalization group. If the relevant (or mass) parameters are taken to zero, the theory approaches the IR fixed point along a RG trajectory. With non-vanishing masses, the theory is driven away from the fixed point. The effects of irrelevant and marginal parameters become smaller along a RG trajectory: the effect of an irrelevant parameter, which is of order 1 at the cutoff energy scale Λ, is suppressed by a positive power of $\frac{\mu}{\Lambda}$ at an arbitrary low energy scale μ, but the effect of a marginal parameter of order 1 at Λ is only suppressed as $\frac{1}{\ln\frac{\Lambda}{\mu}}$ at μ. We cannot take a true continuum limit of this kind of theories; as we take $\Lambda \to \infty$, the effects of marginal parameters vanish, and we are left with a free massive theory.[2] This is called "triviality" in the literature. (See Fig. 1.)

In contrast the other class of theories, QCD-like or asymptotic free theories, admit a true continuum limit. The fixed point of a QCD-like theory is an UV fixed point. We take the relevant parameters to zero as we take the continuum limit. The continuum limit obtained this way is well-defined at all energy scales, and it is parameterized by mass and relevant marginal parameters which drive the theory away from the fixed point along the RG

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[2]We assume that the fixed point is a free massless theory.

trajectories. With a cutoff large but finite, the theory differs from the continuum limit due to the effects of irrelevant parameters, which are suppressed by positive powers of $\frac{\mu}{\Lambda}$.
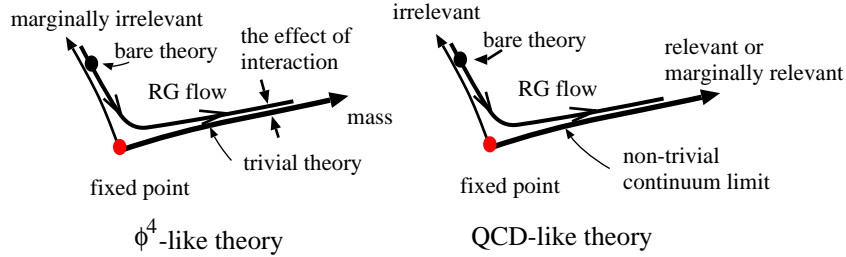

Fig. 1 RG flows

Thus, any theory has good low energy behaviors as long as we fine-tune the mass parameters. Fine-tuning is necessary, since the natural mass scale is Λ, and the mass parameter must be fine-tuned, typically to the order of $\frac{\mu^2}{\Lambda^2}$, to attain a finite physical mass much smaller than the cutoff.[3] A theory is either QCD-like or ϕ^4-like: if it is not QCD-like, it is ϕ^4-like, and with a large but finite cutoff Λ and fine-tuning of mass parameters the theory describes interactions of order $\frac{1}{\ln \frac{\Lambda}{\mu}}$ at a low energy scale μ.

The above RG viewpoint was clearly stated in the first of refs. [2] regarding the equivalence between the non-renormalizable Nambu-Jona-Lasinio (NJL) model and the renormalizable Yukawa theory. Despite their difference in appearance, the two theories are both ϕ^4-like, and they describe the same physics if we ignore the irrelevant differences suppressed by positive powers of $\frac{\mu}{\Lambda}$. Similarly, the $O(N)$ linear- and non-linear sigma models are both ϕ^4-like, and they are equivalent up to differences suppressed by positive powers of $\frac{\mu}{\Lambda}$.[3] As these two concrete examples show, the appearance of a theory at the cutoff scale is misleading. It is the fixed point of the RG which dictates the low energy behaviors of a theory.

The purpose of the present paper is to extend the work of refs. [2] and apply the above RG viewpoint to explore the possibility of constructing gauge theories using apparently non-renormalizable lagrangians without elementary gauge fields. The dynamical generation of gauge symmetries have already been discussed in the literature. Immediately after the work on the NJL model (the first of refs. [2]), purely fermionic construction of QCD was

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[3]In pure QCD we only need to tune but not fine-tune the gauge coupling constant to zero, since it is only marginally relevant. With massive quarks, the quark masses must be fine-tuned to zero to the order $\frac{\mu}{\Lambda}$.

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attempted in ref. [4]. Even earlier, in generalizing the idea of non-linear realizations of symmetries, dynamical generation of gauge symmetries, called hidden local symmetries, was shown to be possible.[5] A non-perturbative study of QCD constructed as an induced gauge theory has been also given in ref. [6].

Our work differs from these earlier works in two aspects: first we will study ϕ^4-like gauge theories from the RG viewpoint given above. We are not introducing new theories. Rather we are showing that certain non-renormalizable theories, often perceived as undesirable, are really the same as perturbatively renormalizable gauge theories. Second we give a detailed analysis of the Ward identities. A renormalizable theory with a vector field is not necessarily a gauge theory. To be a gauge theory, Ward identities must be satisfied. We will show it possible to tune marginally irrelevant parameters to satisfy the necessary Ward identities. In this paper we only consider two theories with abelian gauge symmetries.

A comment is in order regarding the use of $1/N$ expansions in our work. We emphasize that all our results are supported by the RG viewpoint given above, and that they are valid for any N starting from 1. We use the $1/N$ expansions not to study the theories non-perturbatively. We are only interested in perturbation theory with respect to small coupling constants such as the fine structure constant and scalar self-coupling. For ϕ^4-like theories, the so-called non-perturbative effects are all cutoff dependent: their contributions are suppressed by positive powers of $\frac{\mu}{\Lambda}$ which we ignore in our study of non-renormalizable theories. Here we use the $\frac{1}{N}$ expansions to get a small coupling constant of order $\frac{1}{\ln\frac{\Lambda}{\mu}}$ from loop corrections. Naïve perturbative expansions in powers of a bare coupling does not work, since the coupling is of order 1.

This paper is organized as follows. In sect. 2 we review the equivalence of the $O(N)$ non-linear sigma model with the linear sigma model at energies much below the cutoff. This is to remind the reader how misleading the appearance of a lagrangian can be and to emphasize the usefulness of the $1/N$ expansions. In sect. 3 we study a non-renormalizable fermionic model with a current-current interaction and show its equivalence to the standard (massive) QED. This is followed by the study of a little more non-trivial scalar model which also has a current-current interaction in sect. 4. The model is shown to be equivalent to the (massive) scalar QED. Finally, the paper is concluded in sect. 5. For the convenience of the reader, we summarize the Ward identities of QED and scalar QED in appendix A. A table of integrals with a momentum cutoff is given in appendix B.

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Throughout the entire paper we will work in the four dimensional euclidean space. Our convention is that the weight of a euclidean functional integral is given by $\exp[-S] = \exp[-\int d^4x \, \mathcal{L}]$ where S is a euclidean action, and $\mathcal{L}$ is a euclidean action density.[4]

2 Review of the $O(N)$ non-linear sigma model

We begin with a brief review of the equivalence between the $O(N)$ linear and non-linear sigma models.[3] We wish to explain quantitatively how the apparently non-renormalizable non-linear sigma model can be physically equivalent to the perturbatively renormalizable linear sigma model. We first summarize the relevant results on the linear sigma model.[7] The model is defined for N real scalar fields ϕ^I $(I = 1, ..., N)$ by the following action density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^I\partial_\mu\phi^I + \frac{m^2}{2}\phi^I\phi^I + \frac{\lambda}{8N}\left(\phi^I\phi^I\right)^2 \tag{1}$$

with a momentum cutoff Λ_{L}. To leading order in $\frac{1}{N}$, the renormalized squared mass m_r^2 and the renormalized self-coupling λ_r are obtained as[5]

$$m^2 = m_r^2 + \frac{\lambda}{2(4\pi)^2}\left(-\Lambda_{\mathrm{L}}^2 + m_r^2\ln\frac{\Lambda_{\mathrm{L}}^2}{m_r^2}\right) \tag{2}$$

$$\frac{1}{\lambda_r} = \frac{1}{\lambda} + \frac{1}{2(4\pi)^2}\ln\frac{\Lambda_{\mathrm{L}}^2}{\mu^2} \equiv \frac{1}{2(4\pi)^2}\ln\frac{\Lambda_0^2}{\mu^2} \tag{3}$$

where μ is an arbitrary renormalization scale. The correlation functions of ϕ^I are made UV finite in terms of m_r^2 and λ_r. This is the standard renormalization. In Eq. (3) we have introduced the Landau scale Λ_0 at which the bare coupling λ diverges for a given λ_r. The Landau scale gives the largest energy scale beyond which the theory is not defined. The linear sigma model is thus characterized uniquely by two parameters: m_r^2 and Λ_0.

Let us now take a look at the non-linear sigma model. It is defined by the action density

$$\mathcal{L} = N\left[\frac{v^2}{2}\partial_\mu\Phi^I\partial_\mu\Phi^I + \frac{a}{8}\left(\partial_\mu\Phi^I\partial_\mu\Phi^I\right)^2\right] \tag{4}$$

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[4]We call it an action density rather than a lagrangian density to avoid potential confusion about signs.

[5]The terms of order $\frac{m_r^2}{\Lambda_{\mathrm{L}}^2}$ and less are ignored.

with a momentum cutoff Λ_{NL}. Due to the non-linear constraint

$$\Phi^I \Phi^I = 1, \tag{5}$$

the theory is not renormalizable by the usual power counting.

Some comments on the four-derivative term in Eq. (4) are in order. At first sight it seems totally irrelevant; if we expand the action density naïvely in terms of the unconstrained fields $\pi^i = \sqrt{N} v \Phi^i$ ($i = 1, ..., N-1$), the four-derivative term gives rise to an interaction term suppressed by $\frac{1}{v^2}$ which is of order $\frac{1}{\Lambda_{\text{NL}}^2}$. The four-derivative term is indeed irrelevant but not for that reason; it can give marginal contributions, i.e., contributions not suppressed by inverse powers of the momentum cutoff, at low energies. Here, the role of a is merely to rescale the momentum cutoff, and therefore it is physically irrelevant.

It is the existence of a critical value v_c^2 which is the key to the renormalizability of the non-linear model and its equivalence to the linear model. At the critical point $v^2 = v_c^2$ the theory becomes a theory of N free massless scalars. For $v^2 < v_c^2$ the larger fluctuations of the fields are encouraged, and the $O(N)$ symmetry is fully restored. For $v^2 > v_c^2$, however, the fluctuations are discouraged, and the $O(N)$ is spontaneously broken to $O(N-1)$. By fine-tuning the parameter v^2 near v_c^2, the non-linear sigma model gives the same physics as the linear sigma model; all the differences are suppressed by positive powers of $\frac{\mu^2}{\Lambda_{\text{NL}}^2}$ where μ is an arbitrary but finite renormalization scale.

The large N calculations for the non-linear model is well known.[3] For simplicity, we restrict ourselves to the symmetric phase. To leading order in $\frac{1}{N}$, the critical value v_c^2 is given by

$$\frac{v_c^2}{z} \equiv \frac{\Lambda_{\text{NL}}^2}{(4\pi)^2}, \tag{6}$$

where

$$z \equiv 1 - (4\pi)^2 \frac{a}{4}. \tag{7}$$

We take $a < \frac{4}{(4\pi)^2}$ so that $z > 0$.

By straightforward calculations, we can verify that the non-linear model is equivalent to the linear sigma model with renormalized squared mass m_r^2 and self-coupling λ_r, if we choose v^2 by

$$\frac{v^2 - v_c^2}{z} = -\frac{m_r^2}{(4\pi)^2} \ln \frac{\Lambda_{\text{NL}}^2}{m_r^2} < 0 \tag{8}$$

and choose the cutoff $\Lambda_{\rm NL}$ by

$$\ln \frac{\Lambda_{\rm NL}^2}{\Lambda_0^2} = 2(4\pi)^2 \frac{a}{4} \tag{9}$$

where Λ_0 is related to λ_r by Eq. (3). We see that the constant a merely changes the ratio of $\Lambda_{\rm NL}$ to Λ_0 by a finite amount.

A similar analysis can be given to the Nambu-Jona-Lasinio model.[2] The Nambu-Jona-Lasinio model with a non-renormalizable Fermi interaction is equivalent to a perturbatively renormalizable Yukawa theory if we ignore contributions suppressed by negative powers of the momentum cutoff.

3 QED with electrons

In this and next sections we analyze models with non-renormalizable current-current interactions. We first consider a purely fermionic theory defined by the following action density[6]

$$\mathcal{L} = \overline{\psi}^I \left(\frac{1}{i} \slashed{\partial} + iM \right) \psi^I - \frac{1}{2Nv^2} J_\mu J_\mu, \tag{10}$$

where I runs from 1 to N, and the current J_μ is defined by

$$J_\mu \equiv \overline{\psi}^I \gamma_\mu \psi^I. \tag{11}$$

To define a theory we introduce a momentum cutoff Λ. It is essential to use a momentum cutoff as opposed to the dimensional regularization. We are interested in the dependence of the theory on the UV cutoff, and the dimensional regularization is not suitable for this purpose, since it automatically gives the limit of an infinite cutoff.

It turns out that the theory defined by the action density (10) and the current (11) is missing one marginal parameter.[7] Instead of the current given by Eq. (11), we consider a more general

$$J_\mu \equiv \overline{\psi}^I \gamma_\mu \psi^I + \frac{h}{\Lambda^2} \, \overline{\psi}^I \overleftarrow{\slashed{\partial}} \gamma_\mu \slashed{\partial} \psi^I \tag{12}$$

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[6] Our convention for euclidean fermionic fields might differ somewhat from the standard convention. Replace $\overline{\psi}$ by $i\overline{\psi}$ to get a more familiar kinetic term $\overline{\psi}(\slashed{\partial} - M)\psi$. The hermitian gamma matrices γ_μ satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ as usual.

[7] Similarly, in ref. [2] a dimension eight field was introduced to the action density to account for a missing marginal parameter in the naïve NJL model.

The momentum cutoff does not respect gauge invariance, and we need to adjust the coefficient h for the Ward identities.[8]

It is important to observe that the action density $\mathcal{L}$ is invariant under the charge conjugation $\mathcal{C}$ defined by

$$\psi \to C\overline{\psi}^T, \quad \overline{\psi} \to -\psi^T C^{-1} \tag{13}$$

where the four-by-four matrix C satisfies

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T. \tag{14}$$

The current J_μ is odd under $\mathcal{C}$.

It is not necessary but helps our calculations to introduce a vector auxiliary field A_μ. We then rewrite the action density as

$$\begin{aligned}
\mathcal{L} &= \overline{\psi}^I \left(\frac{1}{i}\slashed{\partial} + iM\right)\psi^I - \frac{1}{2Nv^2}J_\mu J_\mu + \frac{1}{2}\left(vA_\mu + \frac{1}{\sqrt{N}v}J_\mu\right)^2 \\
&= \overline{\psi}^I \left(\frac{1}{i}\slashed{\partial} + iM\right)\psi^I + \frac{1}{2}v^2 A_\mu^2 \\
&\quad + \frac{1}{\sqrt{N}}A_\mu\left(\overline{\psi}^I \gamma_\mu \psi^I + \frac{h}{\Lambda^2}\,\overline{\psi}^I \overleftarrow{\slashed{\partial}}\gamma_\mu \slashed{\partial}\psi^I\right)
\end{aligned} \tag{15}$$

What do we expect at energy scales much below the cutoff Λ? We ignore the effects suppressed by the inverse powers of the cutoff Λ but keep those only suppressed by negative powers of the logarithm of Λ. If we fine-tune the mass parameter v so that the mass scale of the theory remains UV finite, we expect that the above theory becomes equivalent to a theory which is renormalizable by power counting. The smallest renormalizable theory with fermions ψ^I and a real vector field A_μ is given by the following action density:

$$\mathcal{L}_{ren} = \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi_0}(\partial_\mu A_\mu)^2 + \frac{m_0^2}{2}A_\mu^2 + \frac{\lambda_0}{8N}(A_\mu^2)^2 + \overline{\psi}^I\left(\frac{1}{i}\slashed{\partial} + \frac{e_0}{\sqrt{N}}\slashed{A} + iM\right)\psi^I \tag{16}$$

where we have imposed the $\mathcal{C}$ invariance. At tree level the Ward identity demands that the parameter λ_0 vanish. But renormalizability alone allows an arbitrary λ_0.

To leading order in $\frac{1}{N}$ the fermion-photon vertex receives no radiative correction, and we only need to calculate the one-loop contributions to the two- and four-point proper vertices of the photon field A_μ. The three-point vertex vanishes due to the $\mathcal{C}$ invariance. If the theories defined by (15)

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[8]See Appendix A for a summary of the Ward identities for QED.

and (16) have the same two- and four-point vertices, the two theories are equivalent, since any higher point functions can be constructed out of the fermion-photon vertex and the photon two- and four-point vertices independently of the cutoff Λ.

The one-loop calculations with a momentum cutoff Λ are straightforward, and we only write the results here. The inverse propagator is calculated from the top one-loop diagram in Fig. 2 as follows:

$$\Pi_{\alpha\beta}(k^2) = \frac{1}{e^2}\left[m_\gamma^2 \delta_{\alpha\beta} + \frac{1}{\xi} k_\alpha k_\beta \right. \tag{17}$$

$$\left. +(k^2\delta_{\alpha\beta} - k_\alpha k_\beta)\left(1 - 8\frac{e^2}{(4\pi)^2}\int_0^1 dx\ x(1-x)\ln\frac{M^2 + x(1-x)k^2}{\mu^2} \right) \right]$$

where we define

$$\frac{(4\pi)^2}{e^2} \equiv \frac{4}{3}\ln\frac{\Lambda^2}{\mu^2} - 1 + \frac{4}{3}h - \frac{1}{9}h^2 \tag{18}$$

$$\frac{(4\pi)^2}{e^2}m_\gamma^2 \equiv (4\pi)^2 v^2 + \left(-2 + 4h - \frac{2}{3}h^2\right)\Lambda^2 + (2 - 12h)M^2 \tag{19}$$

$$\frac{(4\pi)^2}{e^2}\frac{1}{\xi} \equiv \frac{1}{3}(1 + 4h - h^2) \tag{20}$$

Eq. (19) implies that we must fine-tune the squared mass parameter v^2 so that the squared mass m_γ^2 of the photon is finite and positive. This is fine-tuning as opposed to tuning, since $\frac{v^2}{\Lambda^2}$ must be tuned to an order 1 quantity to the accuracy of $\frac{M^2}{\Lambda^2}$. With this fine-tuning, the above photon two-point function is identical to the one in the massive QED with the running gauge coupling constant e, photon mass m_γ, and gauge fixing parameter ξ. The mass parameter μ is an arbitrary renormalization scale.

Fig. 2   Leading contributions in 1/N

The four-point function at zero external momenta is obtained from the

bottom one-loop graphs in Fig. 2, and it depends on h:

$$\Pi_{\alpha\beta\gamma\delta} = \frac{1}{N} \frac{1}{(4\pi)^2} \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} \right) \left(\frac{4}{3} - 16h + 24h^2 - \frac{16}{3}h^3 \right).$$
(21)

For the theory to be the massive QED, the Ward identity for the four-point function must be satisfied. Hence, we must choose the parameter h such that the above four-point function vanishes:

$$\frac{4}{3} - 16h + 24h^2 - \frac{16}{3}h^3 = 0.$$
(22)

This equation has three real roots. We can choose, for example, $h \simeq 0.097$. We note that the choice of h is a tuning, but not a fine-tuning; it must be tuned relative to order 1, but not $\frac{\mu^2}{\Lambda^2}$.

The result corresponding to $m_\gamma = 0$ can be obtained from the induced QED which is defined by the action density (15) with a specific choice $v = h = 0$.[9] We have chosen to study the more general (15) because our interest is to verify the equivalence of the non-renormalizable theory to the renormalizable theory which is defined by Eq. (16) with arbitrary parameters m_0^2, λ_0.

We have chosen the interaction to have the current-current form, but it was not necessary. Instead of modifying the current by a dimension 5 term proportional to h, we could have introduced a counterterm

$$\text{const} \times \left(\frac{1}{N\Lambda^4} \left(\overline{\psi}^I \gamma_\mu \psi^I \right)^2 \right)^2$$
(23)

in the action density. This is how we introduce missing marginal parameters for scalar QED in the next section.

Before closing this section, we make a remark on the sign of the interaction term in the action density (10). Since the current J_μ defined by Eq. (12) is a real field, the current-current interaction term in (10) is negative definite. We believe it does not invalidate the theory. Our reasoning goes as follows. First we note that the theory defined by (10) is equivalent to the massive QED, which is a stable theory, modulo irrelevant differences of order $\frac{\mu^2}{\Lambda^2}$. Therefore, if the theory is unstable, the effects of the instability must be suppressed by positive powers of $\frac{\mu^2}{\Lambda^2}$. This suggests that a potential instability can only arise from the large fluctuations of the fields,

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[9]In addition we must use a cutoff which respects the gauge invariance. Otherwise the electron loops will not be gauge invariant.

for example A_μ of order Λ. If this is the case, stability will be assured by redefining the theory by the action density (15) where the auxiliary field A_μ is restricted within a finite range $|A_\mu| < \Lambda$. The effects of this modification are suppressed by positive powers of $\frac{\mu^2}{\Lambda^2}$.

4   Scalar QED

The fermionic theory may be a little too simple. It resembles the induced QED too much. If we had used a regularization which allows shifts of momentum such as the dimensional regularization, the vacuum polarization would have come out transverse, and the photon four-point function would have vanished at zero external momenta. The Ward identities are then satisfied automatically. Let us introduce a more non-trivial example of a purely bosonic theory in this section.

The theory is defined by the following action density:

$$
\begin{aligned}
\mathcal{L} &= \partial_\mu \phi^{I*} \partial_\mu \phi^I + m^2 \phi^{I*} \phi^I + \frac{\lambda}{4N} \left(\phi^{I*} \phi^I \right)^2 \\
&\quad - \frac{v^2}{2} \left(\frac{-i}{\sqrt{N} v^2} \phi^{I*} \overleftrightarrow{\partial_\mu} \phi^I \right)^2 + \Delta\mathcal{L},
\end{aligned}
\tag{24}
$$

where the asterisk $*$ denotes complex conjugation, and the counterterm $\Delta\mathcal{L}$ is defined by

$$
\begin{aligned}
\Delta\mathcal{L} &\equiv \frac{a}{N} (\phi^{I*} \phi^I) \frac{1}{2} \left(\frac{-i}{\sqrt{N} v^2} \phi^{I*} \overleftrightarrow{\partial_\mu} \phi^I \right)^2 \\
&\quad + \frac{b}{N(4\pi)^2} \frac{1}{8} \left\{ \left(\frac{-i}{\sqrt{N} v^2} \phi^{I*} \overleftrightarrow{\partial_\mu} \phi^I \right)^2 \right\}^2.
\end{aligned}
\tag{25}
$$

We have introduced enough number of parameters so that the theory is equivalent to the theory defined by the following renormalizable action density:

$$
\begin{aligned}
\mathcal{L}_{ren} &= \partial_\mu \phi^{I*} \partial_\mu \phi^I + m_0^2 \phi^{I*} \phi^I + \frac{\lambda_0}{4N} (\phi^{I*} \phi^I)^2 \\
&\quad + \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\xi_0} (\partial_\mu A_\mu)^2 + \frac{m_{\gamma,0}^2}{2} A_\mu^2 \\
&\quad + \frac{e_0}{\sqrt{N}} A_\mu i \phi^{I*} \overleftrightarrow{\partial_\mu} \phi^I + \frac{\gamma}{2N} \phi^{I*} \phi^I A_\mu^2 + \frac{\delta}{8N} \frac{1}{(4\pi)^2} (A_\mu^2)^2
\end{aligned}
\tag{26}
$$

Just as the Ward identities can determine the constants γ and δ uniquely, we will be able to fix the coefficients a, b of the counterterms by imposing the

Ward identities. We should note that unlike the fermionic theory discussed in the previous section, the interaction of this theory is not solely current-current type due to the counterterms.

To facilitate the $\frac{1}{N}$ expansions, we introduce scalar and vector auxiliary fields α, A_μ and rewrite the action density as follows:

$$
\begin{aligned}
\mathcal{L} &= \partial_\mu \phi^{I*} \partial_\mu \phi^I + m^2 \phi^{I*} \phi^I + \frac{\lambda}{4N} \left(\phi^{I*} \phi^I \right)^2 - \frac{v^2}{2} \left(\frac{-i}{\sqrt{N} v^2} \phi^{I*} \overset{\leftrightarrow}{\partial}_\mu \phi^I \right)^2 \\
&\quad + \frac{1}{2} \left(\alpha + i \sqrt{\frac{\lambda}{N}} \phi^{I*} \phi^I \right)^2 + \frac{1}{2} \left(v A_\mu + \frac{i}{\sqrt{N} v} \phi^{I*} \overset{\leftrightarrow}{\partial}_\mu \phi^I \right)^2 + \Delta \mathcal{L} \\
&= \partial_\mu \phi^{I*} \partial_\mu \phi^I + \frac{1}{2} \alpha^2 + i \sqrt{\frac{\lambda}{N}} \, \alpha \, \phi^{I*} \phi^I \\
&\quad + \frac{v^2}{2} A_\mu^2 + \frac{1}{\sqrt{N}} A_\mu i \phi^{I*} \overset{\leftrightarrow}{\partial}_\mu \phi^I + \Delta \mathcal{L}.
\end{aligned}
\tag{27}
$$

Unlike the fermionic theory of the previous section, this theory does not reduce to an induced QED for $v = 0$.

To leading order in $\frac{1}{N}$, we must renormalize the scalar mass and self-coupling as

$$
\Delta m^2 \equiv m_r^2 - m^2 = \frac{\lambda}{(4\pi)^2} \left(\Lambda^2 - m_r^2 \ln \frac{\Lambda^2}{m_r^2} \right)
\tag{28}
$$

$$
\frac{(4\pi)^2}{\lambda_r} = \frac{(4\pi)^2}{\lambda} + \ln \frac{\Lambda^2}{\mu^2} - 1
\tag{29}
$$

and shift the auxiliary field α by

$$
\alpha = -i \sqrt{\frac{N}{\lambda}} \, \Delta m^2 + \sqrt{\frac{\lambda_r}{\lambda}} \, \delta \alpha,
\tag{30}
$$

where the shifted field $\delta \alpha$ has a vanishing expectation value $\langle \delta \alpha \rangle = 0$. To leading order in $\frac{1}{N}$, the full propagator of the fluctuation $\delta \alpha$ is given by

$$
\left\langle \widetilde{\delta \alpha}(k) \delta \alpha \right\rangle = 1 \bigg/ \left(1 + \frac{\lambda_r}{(4\pi)^2} \int_0^1 dx \, \ln \frac{\mu^2}{m_r^2 + x(1-x)k^2} \right).
\tag{31}
$$

Before calculating the vertex functions involving the vector field A_μ, we note the implication of the equation of motion for A_μ. The action density is quadratic with respect to A_μ, and the equation of motion gives

$$
A_\mu = B_\mu \equiv \frac{-i}{\sqrt{N} v^2} \phi^{I*} \overset{\leftrightarrow}{\partial}_\mu \phi^I
\tag{32}
$$

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This implies that the composite field B_μ is an interpolating field of the photon. The calculation of the proper vertex of A_μ and B_ν indeed gives

$$\left\langle \widetilde{A_\mu}(k)B_\nu \right\rangle = \delta_{\mu\nu} \frac{1}{v^2} \int_p \frac{p^2}{(p^2 + m_r^2)^2} \simeq \delta_{\mu\nu} \frac{1}{v^2} \frac{\Lambda^2}{(4\pi)^2} \qquad (33)$$

where we have ignored the terms of order $\frac{m_r^2}{\Lambda^2}$. (Fig. 3) This is $\delta_{\mu\nu}$ if we choose

$$v^2 \simeq \frac{\Lambda^2}{(4\pi)^2}. \qquad (34)$$

We will see that Eq. (34) is required by the fine-tuning of the mass parameter. Hence, the equation of motion (32) implies that the counterterm in the action density is equivalent to

$$\Delta\mathcal{L} = \frac{a}{2N} \left(\phi^{I*}\phi^I \right) A_\mu^2 + \frac{b}{8N}(A_\mu^2)^2 \qquad (35)$$

to leading order in $\frac{1}{N}$. Therefore, $\Delta\mathcal{L}$ gives rise to the vertices in Fig. 4.
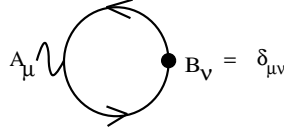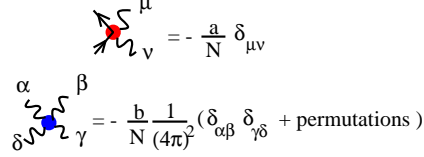

Fig. 3 Two-point vertex of A_μ and B_ν Fig. 4 Counterterms

Let us proceed with the two-point vertex of the photon. To leading order in $\frac{1}{N}$, we obtain

$$\Pi_{\alpha\beta} = \frac{1}{e^2} \left[m_\gamma^2 \delta_{\alpha\beta} + \frac{1}{\xi} k_\alpha k_\beta \right. \qquad (36)$$

$$\left. +(k^2\delta_{\alpha\beta} - k_\alpha k_\beta)\left(1 - \frac{e^2}{(4\pi)^2} \int_0^1 dx(1-2x)^2 \ln \frac{m_r^2 + x(1-x)k^2}{\mu^2} \right) \right],$$

where the renormalized parameters are defined by

$$\frac{(4\pi)^2}{e^2} \equiv \frac{1}{3}\ln\frac{\Lambda^2}{\mu^2} - \frac{1}{2} \qquad (37)$$

$$m_\gamma^2 \equiv e^2\left[v^2 - \frac{1}{(4\pi)^2}\left(\Lambda^2 + m_r^2 - 2m_r^2\ln\frac{\Lambda^2}{m_r^2} \right) \right] \qquad (38)$$

$$\frac{1}{\xi} \equiv -\frac{e^2}{(4\pi)^2}\frac{1}{6}. \qquad (39)$$

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Here, μ is an arbitrary renormalization scale as usual. Eq. (38) implies that we need to fine-tune the squared mass parameter v^2 such that $0 < m_\gamma^2 \ll \Lambda^2$.

Now that we have identified the photon mass m_γ and the gauge coupling e, we wish to proceed with verifying the equivalence of our theory with the scalar QED. This will be done by checking three Ward identities. (See Appendix A for a summary of the Ward identities.) The first is the Ward identity for the three-point vertex of the scalar and photon. To leading order in $\frac{1}{N}$, the vertex receives no radiative correction (Fig. 5), and the Ward identity is automatically satisfied.

Fig. 5   interaction vertex

Next we examine the scalar-scalar-photon-photon vertex. To leading order in $\frac{1}{N}$, it is given by the four diagrams in Fig. 6, where we denote the propagator of $\delta\alpha$ by a broken line.

Fig. 6   Leading contributions to the scalar-scalar-photon-photon vertex

The third and fourth terms involve the counterterm proportional to a. The Ward identity demands that at zero external momenta this be given by

$$ -\Pi_{\mu\nu}^{\phi\phi^*}\Big|_{\text{zero momenta}} = -\frac{2}{N}\, \delta_{\mu\nu}. \tag{40}$$

To leading order in $\frac{1}{N}$ we obtain

$$ -\Pi_{\mu\nu}^{\phi\phi^*}\Big|_{\text{zero momenta}} = -\frac{2}{N}\, \delta_{\mu\nu}\, \frac{\ln\frac{\Lambda^2}{m_r^2} - 1 + \frac{a}{2}\frac{(4\pi)^2}{\lambda} - \frac{1}{2}}{\ln\frac{\Lambda^2}{m_r^2} - 1 + \frac{(4\pi)^2}{\lambda}}. \tag{41}$$

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Therefore, we must choose a as

$$a = \frac{\lambda}{(4\pi)^2} + 2. \tag{42}$$

This is an ordinary tuning, since λ is a quantity of order 1.

Finally we examine the four-photon vertex. Many graphs contribute to leading order in $\frac{1}{N}$ as in Fig. 7.
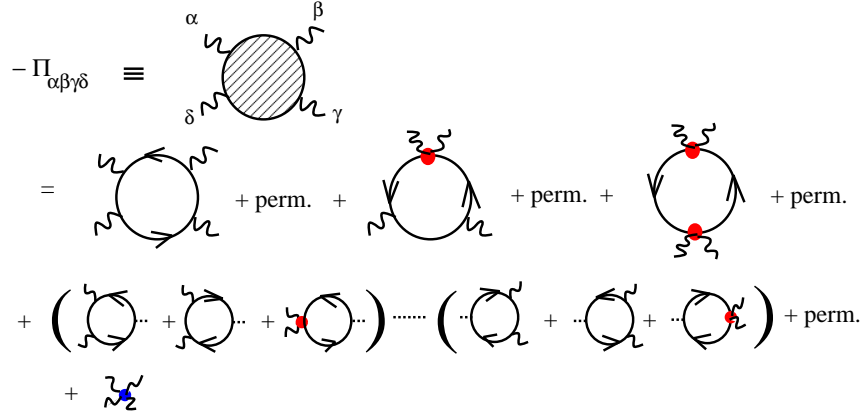

Fig. 7 Four-photon vertex

The last term in Fig. 7 is the counterterm proportional to b. The Ward identity demands that the four-photon vertex vanish for zero external momenta. The calculation is straightforward but somewhat involved. We can use Eq. (40) to simplify the calculation. The final result is given by

$$-\Pi_{\alpha\beta\gamma\delta}\Big|_{\text{zero momenta}} = \frac{1}{N}\frac{1}{(4\pi)^2}\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)\left(-\frac{4}{3} + a - b\right). \tag{43}$$

Therefore, the Ward identity demands

$$b = \frac{\lambda}{(4\pi)^2} + \frac{2}{3} \tag{44}$$

where we have used Eq. (42). Thus, with the choice of the coefficients a, b given by Eqs. (42,44), the theory defined by the action density (24) is equivalent to the scalar QED.

Before we close this section, we make a brief comment on the relation of the above model to the CP^{N-1} model.[8] (See also Chapter 5 of ref. [5].)

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The CP^{N-1} model can be obtained formally from the action density (24) by imposing a non-linear constraint

$$\phi^{I*}\phi^I = \frac{Nv^2}{2}.$$
(45)

Since the $O(2N)$ non-linear sigma model is equivalent to the $O(2N)$ linear sigma model, the CP^{N-1} model is equivalent to the model discussed in this section, and therefore to the scalar QED.

5 Conclusion

We have seen that once the mass parameters are fine-tuned, theories which are non-renormalizable by the usual power counting rule reduce to renormalizable theories at energies below the cutoff Λ as long as we ignore quantities inversely proportional to Λ. The renormalized parameters at a finite energy scale μ appear as the marginally irrelevant dependence on the cutoff of the order of $\frac{1}{\ln\frac{\Lambda}{\mu}}$. We have studied two matter-only models without elementary gauge fields whose current-current contact interactions at the cutoff scale give rise to interactions mediated by the abelian gauge fields at low energies. The gauge field was dynamically generated to complete renormalizability of the theory just as the sigma field is dynamically generated in the $O(N)$ non-linear sigma model.

In this paper the massive photon was obtained not as a consequence of the Higgs mechanism: it appeared as part of the gauge fixing terms. The mass is simply allowed by the Ward identities in the case of abelian gauge theories. In the next paper we will study a matter-only model which exhibits the Higgs mechanism.[9]

It should be interesting to extend our analysis to non-abelian gauge theories. We expect that the covariant gauge fixing term will arise naturally just as for QED, but in the case of non-abelian gauge theories the covariant gauge requires the Faddeev-Popov (FP) ghosts. Hence, for a matter-only model to become a non-abelian gauge theory, the FP ghosts must be generated dynamically. It will be extremely interesting if this is the case.

In ref. [6] an induced QCD was studied in the $1/N$ expansions in the hope of uncovering the non-perturbative dynamics of QCD. We also hope that the reformulation of gauge theories as matter-only theories will find practical applications in understanding, for example, the physics of QED at energies very high compared to the electron mass but still much lower than the cutoff scale.

A   Ward identities

In determining the coefficients of the counterterms, we have imposed Ward identities. We remind the reader of the Ward identities for both QED with electrons and the scalar QED. For QED with electrons, we have three Ward identities to satisfy:

$$k_\mu \Pi_{\mu\nu}(k) = \frac{k_\nu}{e^2}\left(m_\gamma^2 + \frac{1}{\xi}k^2\right) \tag{46}$$

$$-ik_\mu \Pi_\mu^{\psi\overline{\psi}}(p,k) = \frac{i}{\sqrt{N}}\left(\Pi^{\psi\overline{\psi}}(p) - \Pi^{\psi\overline{\psi}}(p+k)\right) \tag{47}$$

$$k_\alpha \Pi_{\alpha\beta\gamma\delta} = 0, \tag{48}$$

where $\Pi_\mu^{\psi\overline{\psi}}(p,k)$ is the electron-photon interaction vertex, and $\Pi^{\psi\overline{\psi}}(p)$ is the inverse electron propagator with momentum p.
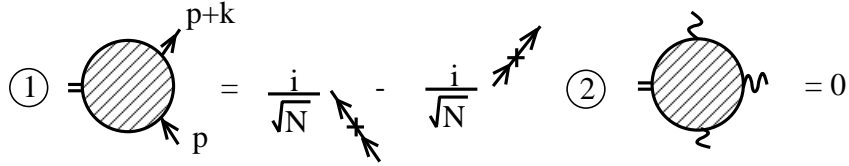

Fig. 8 Ward identities for QED with electrons

The Ward identities for the scalar QED are similarly given by

$$k_\mu \Pi_{\mu\nu}(k) = \frac{k_\nu}{e^2}\left(m_\gamma^2 + \frac{1}{\xi}k^2\right) \tag{49}$$

$$-ik_\mu \Pi_\mu^{\phi\phi^*}(p,k) = \frac{i}{\sqrt{N}}\left(\Pi^{\phi\phi^*}(p) - \Pi^{\phi\phi^*}(p+k)\right) \tag{50}$$

$$-ik_\mu \Pi_{\mu\nu}^{\phi\phi^*}(p,k,l) = \frac{i}{\sqrt{N}}\left(-\Pi_\nu^{\phi\phi^*}(p,l) + \Pi_\nu^{\phi\phi^*}(p+k,l)\right) \tag{51}$$

$$k_\alpha \Pi_{\alpha\beta\gamma\delta} = 0 \tag{52}$$

using a similar notation as for QED.

Fig. 9 Ward identities for scalar QED

B Integrals with a momentum cutoff

In calculating the one-loop integrals with a momentum cutoff, we have used the following formulas where contributions of order $\frac{m^2}{\Lambda^2}$ or less are ignored.

$$
\begin{aligned}
\int_{p<\Lambda} \frac{1}{p^2 + m^2} &\equiv \int_{p^2<\Lambda^2} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} \\
&= \frac{1}{(4\pi)^2} \left(\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right) & (53)
\end{aligned}
$$

$$
\begin{aligned}
\int_{p<\Lambda} \frac{p_\mu p_\nu}{(p^2 + m^2)^2} &= \frac{\delta_{\mu\nu}}{4} \int_p \frac{p^2}{(p^2 + m^2)^2} \\
&= \frac{\delta_{\mu\nu}}{4} \frac{1}{(4\pi)^2} \left(\Lambda^2 - 2m^2 \ln \frac{\Lambda^2}{m^2} + m^2 \right) & (54)
\end{aligned}
$$

$$
\int_{p<\Lambda} \frac{1}{(p^2 + m^2)^2} = \frac{1}{(4\pi)^2} \left(\ln \frac{\Lambda^2}{m^2} - 1 \right) \tag{55}
$$

$$
\int_{p<\Lambda} \frac{p_\mu p_\nu}{(p^2 + m^2)^3} = \frac{\delta_{\mu\nu}}{4} \frac{1}{(4\pi)^2} \left(\ln \frac{\Lambda^2}{m^2} - \frac{3}{2} \right) \tag{56}
$$

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